Graph Theory and Network Science

Girls Talk Math
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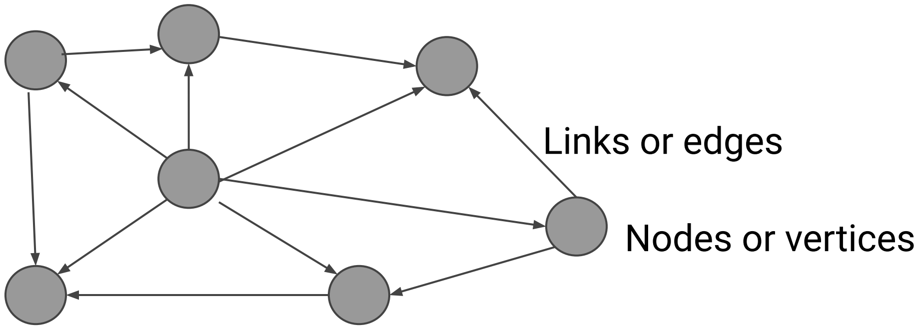
Why are we interested in networks?

- The use of networks is a popular tool for
  - Data representation
  - Organization
  - Analyze complex systems
- These systems usually have a pattern of connections that can be represented as a network.
- It is of interest to scientist to develop a variety of tools to analyze, model and understand these networks.

What is a network?

A network, in its simplest form, is a collection of points joined together in pairs by lines.



Examples of Networks

Technological Networks

- The internet
- Telephone network
- Power grids
- Transportation networks

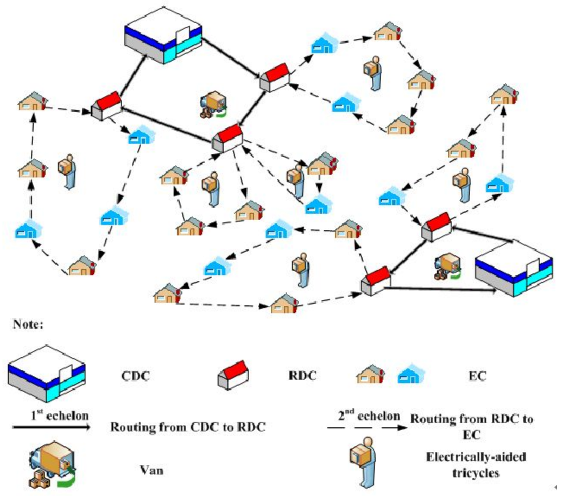
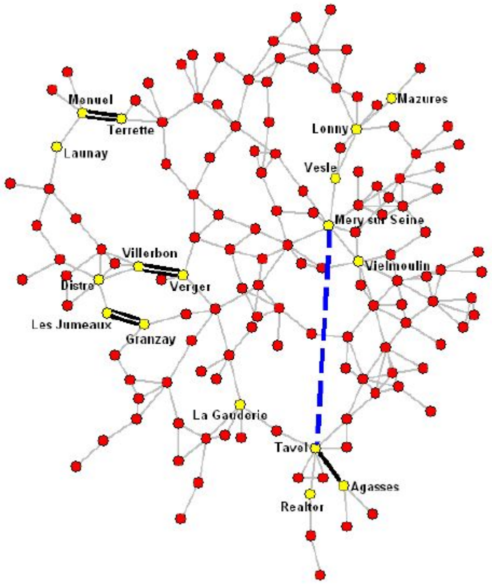


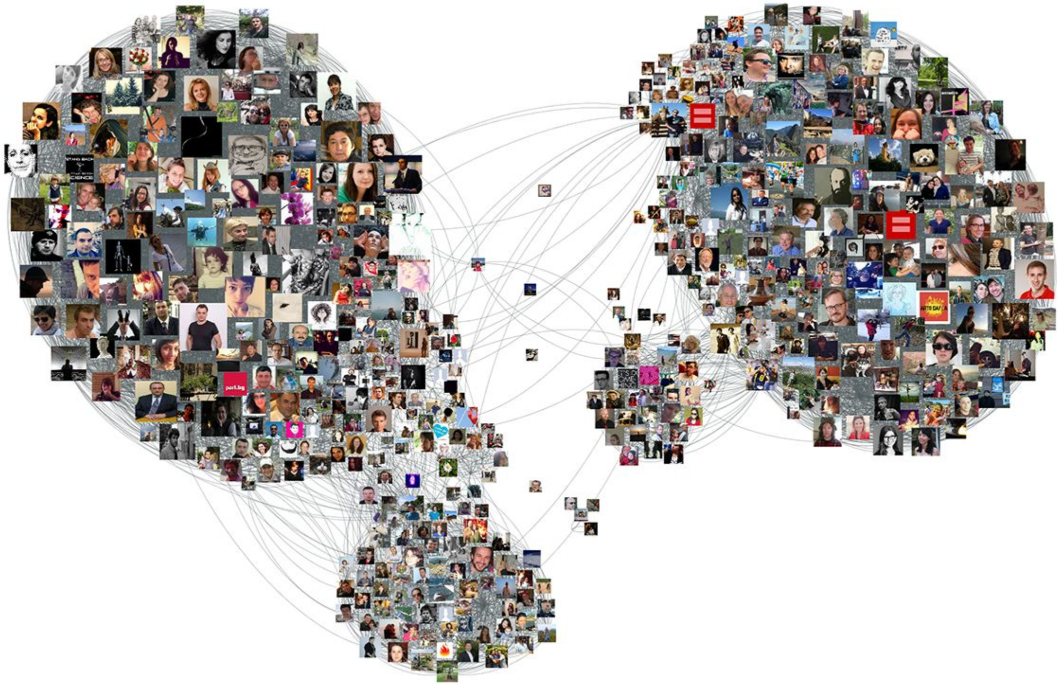
Fig. 2. Distribution network of parcel delivery based on 2E-LRP model



France power grid network

New York subway transit map

Social Networks

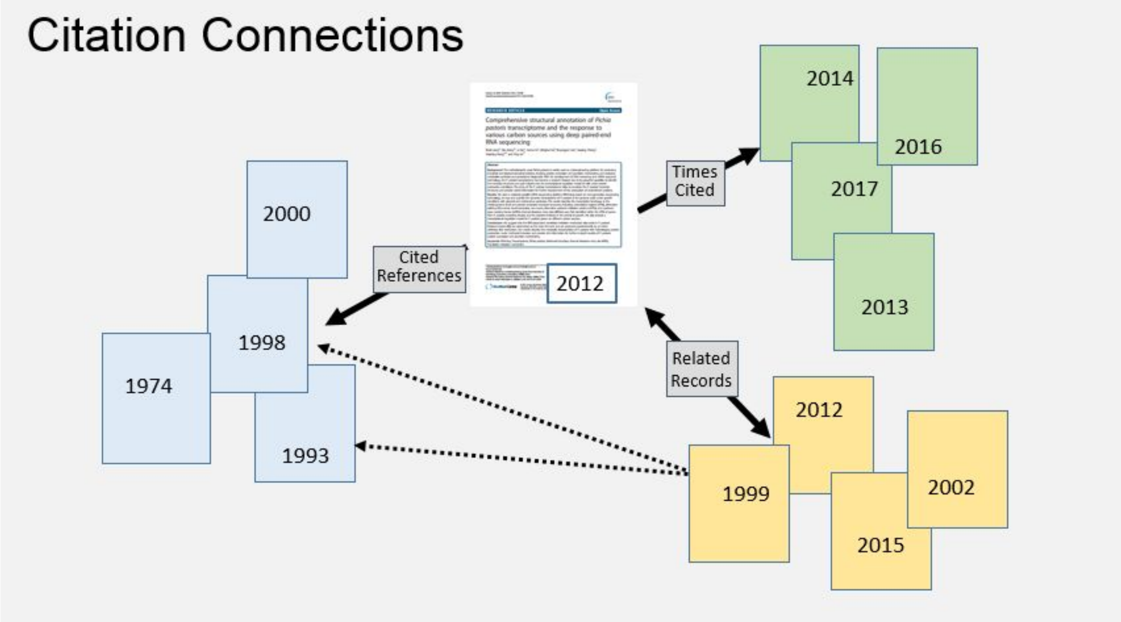


Facebook Network

- Facebook network
- Actor network
- Science collaboration network

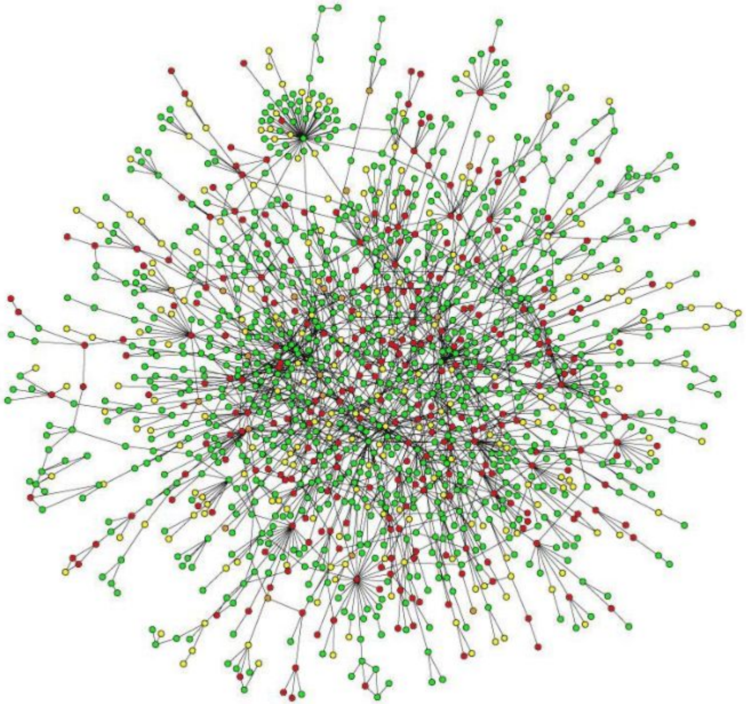
Networks of Information

- The world wide web
- Citation network
- Recommender network

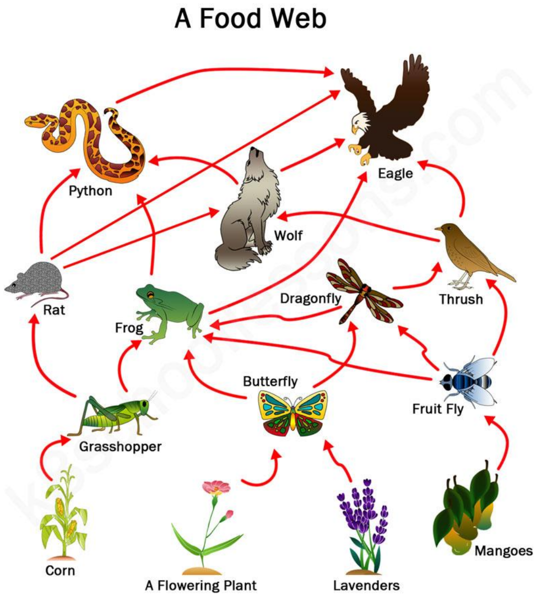


Biological Networks

- Neural networks
- Protein interaction network
- Ecological network



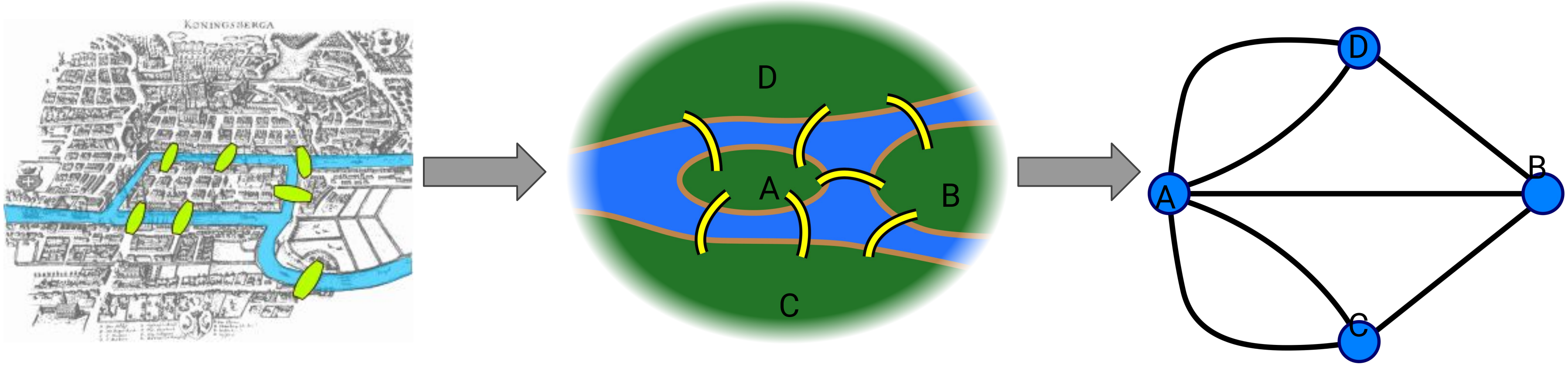
Protein-protein interaction network of yeast



History of Network Science

In the beginning...

The seven bridges of Königsberg is considered the first problem studied in graph theory.



Is it a network or a graph?

Network Science	Graph Theory
Network	Graph
Node	Vertex
Link	Edge

Networks often refer to real systems (WWW is a network of web documents linked by URLs, facebook is a network of users linked by friendships, etc.)

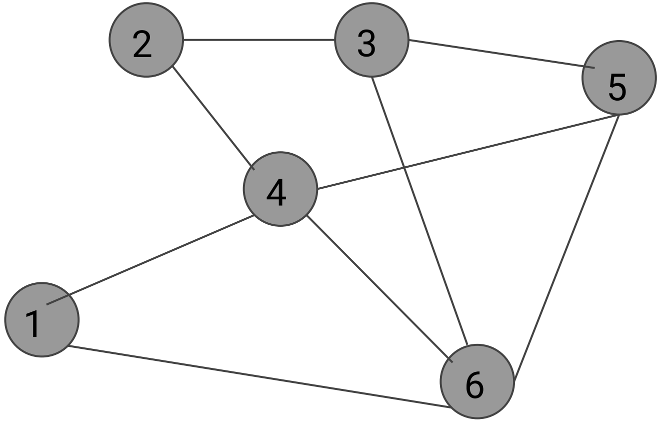
Graphs often refer to the mathematical representation of these networks (web graph, social graph, etc.)

The distinction is rarely made and the two terminologies are often synonyms.

Mathematics of Networks

Degree of a node

The degree of a node is the number of links it has to other nodes. Usually denote k_i as the degree of node i.

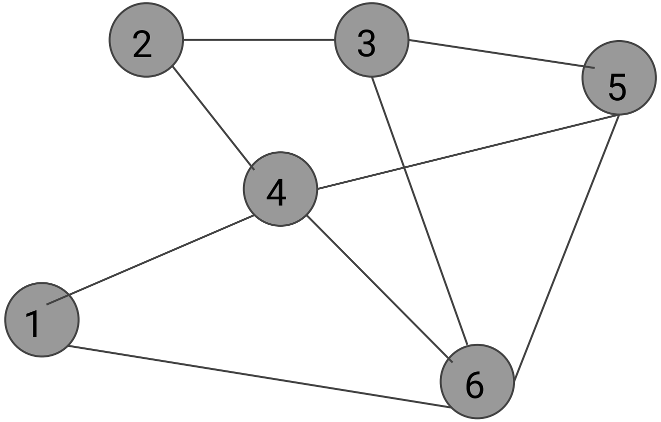


k_4=4

k_5=3

Average Degree

The average degree is the sum of the degree of every node divided by the total number of nodes. The average degree is given by $< k >= \frac{1}{N} \sum_{i=1}^{N} k_i$



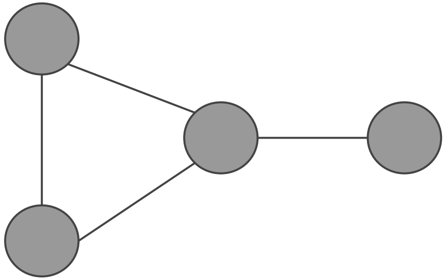
$k_1=2$
$k_2=2$
$k_3=3$
$k_4=4$
$k_5=3$
$k_6=4$

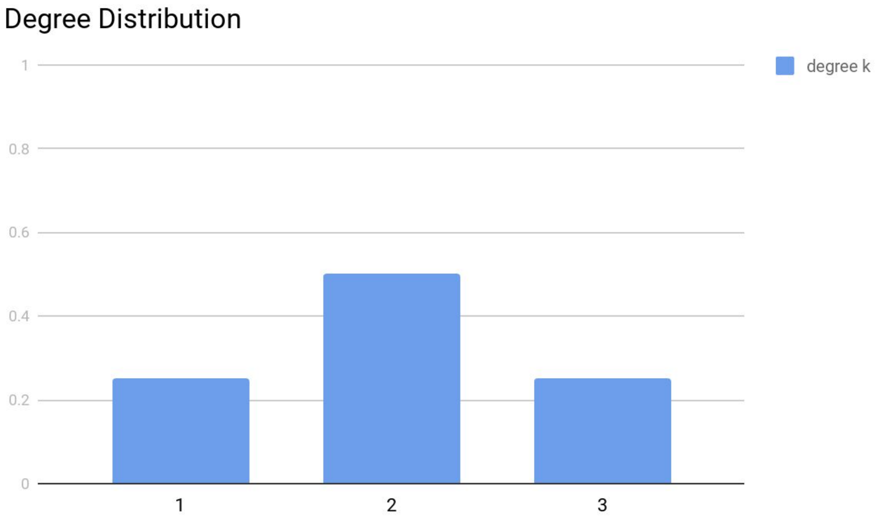
$<k> = 3$

Degree Distribution

Degree distribution, p_k, is the probability that a randomly selected node has degree k. The degree distribution is given by

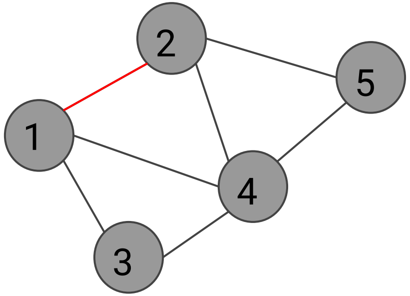
$$p_k = \frac{N_k}{N}$$





Adjacency Matrix

- A matrix is a rectangular array of numbers arranged in rows and columns.
- An adjacency matrix is a way to keep track of the links in the networks. The entry
  - $A_{i,j}$ = 1 if there is an edge from i to j
  - $A_{i,j}$ = 0 otherwise

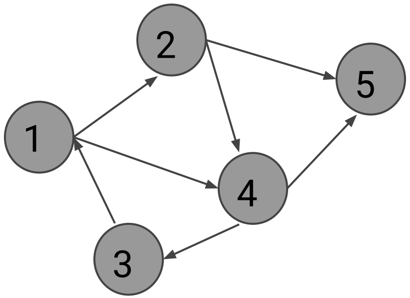


$A_{1,2}$

$$\begin{bmatrix} 0 & \boxed{1} & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Paths and Cycles

- A path from i to j is a sequence of adjacent nodes starting with i and ending with j.
- The number of edges involved in a path is called the length.
- A path that starts and ends at the same node is called a cycle.



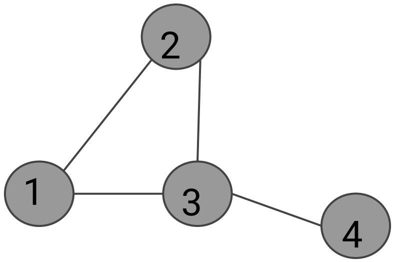
How many paths from node 1 to node 5 can you find?

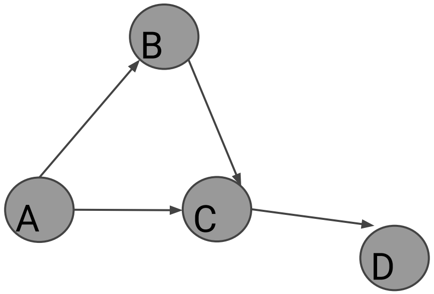
Are there any cycles in the graph?

Types of Networks

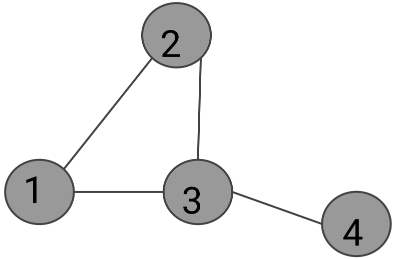
Directed VS Undirected

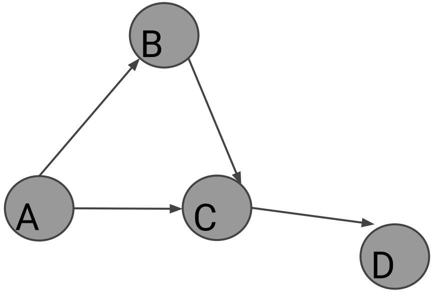
- Networks in which the direction of relationships matter are called *directed* networks.
- Networks in which the direction doesn't matter or the relationships always go both ways are called *undirected* networks.





How is changes the adjacency matrix



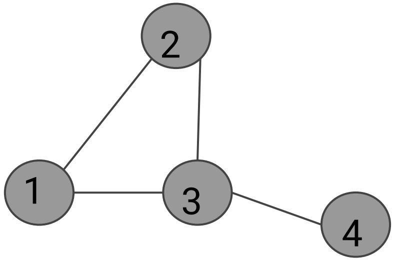


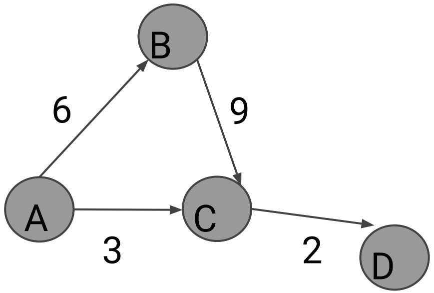
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

	A	B	C	D
A	0	1	1	0
B	0	0	1	0
C	0	0	0	1
D	0	0	0	0

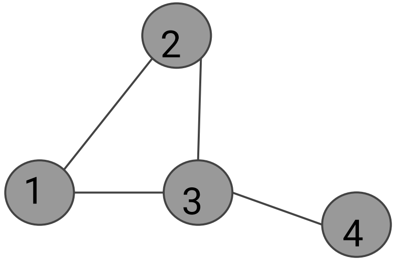
Weighted VS Unweighted

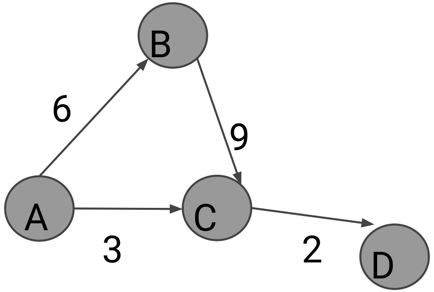
- A network where each edge has a weight is called a *weighted* network.
- Networks without weights are called *unweighted* networks.





How this changes the adjacency matrix



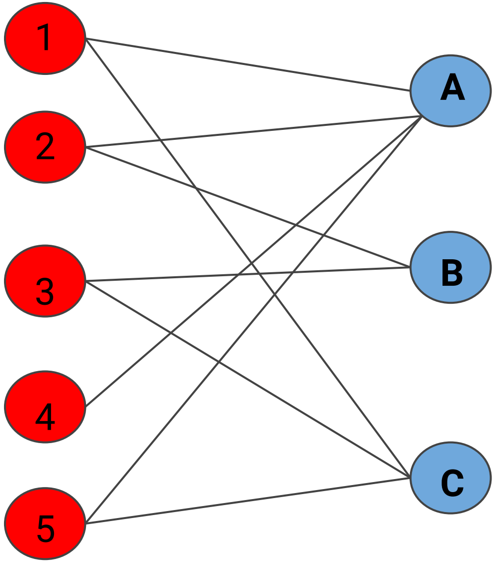


$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

	A	B	C	D
A	0	6	3	0
B	0	0	9	0
C	0	0	0	2
D	0	0	0	0

Bipartite Networks

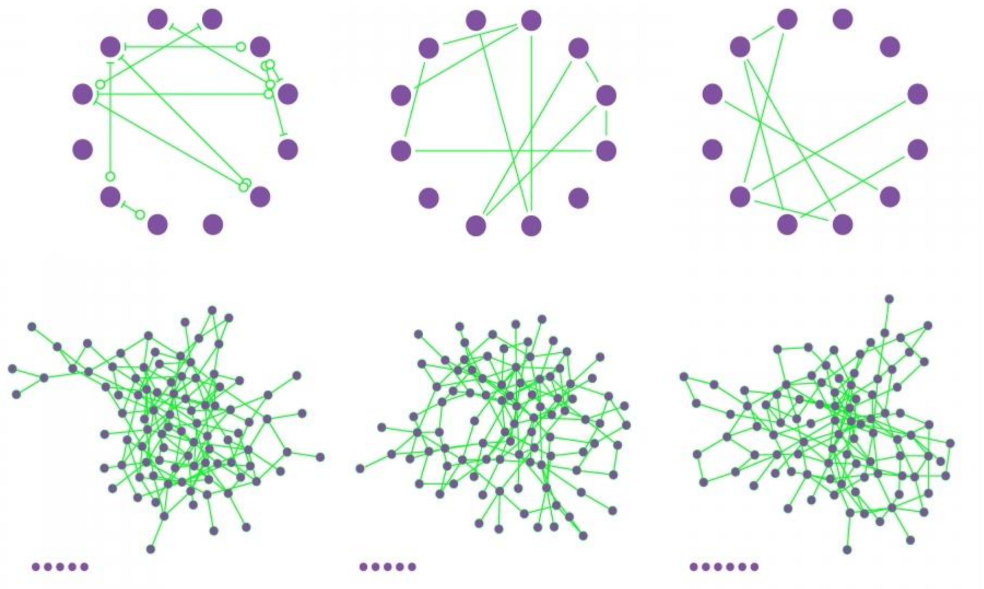
A bipartite graph is a network whose nodes can be divided into two disjoint sets.



Network Models

Random Network Model

A random network consist of N nodes where each node pair is connected with probability p.

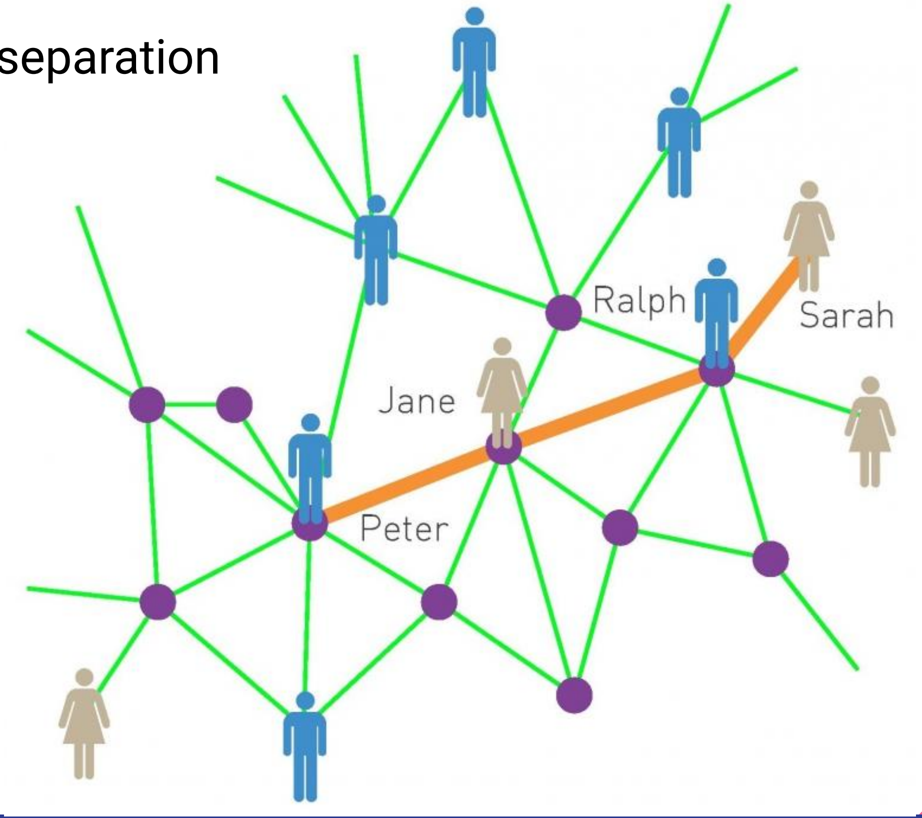


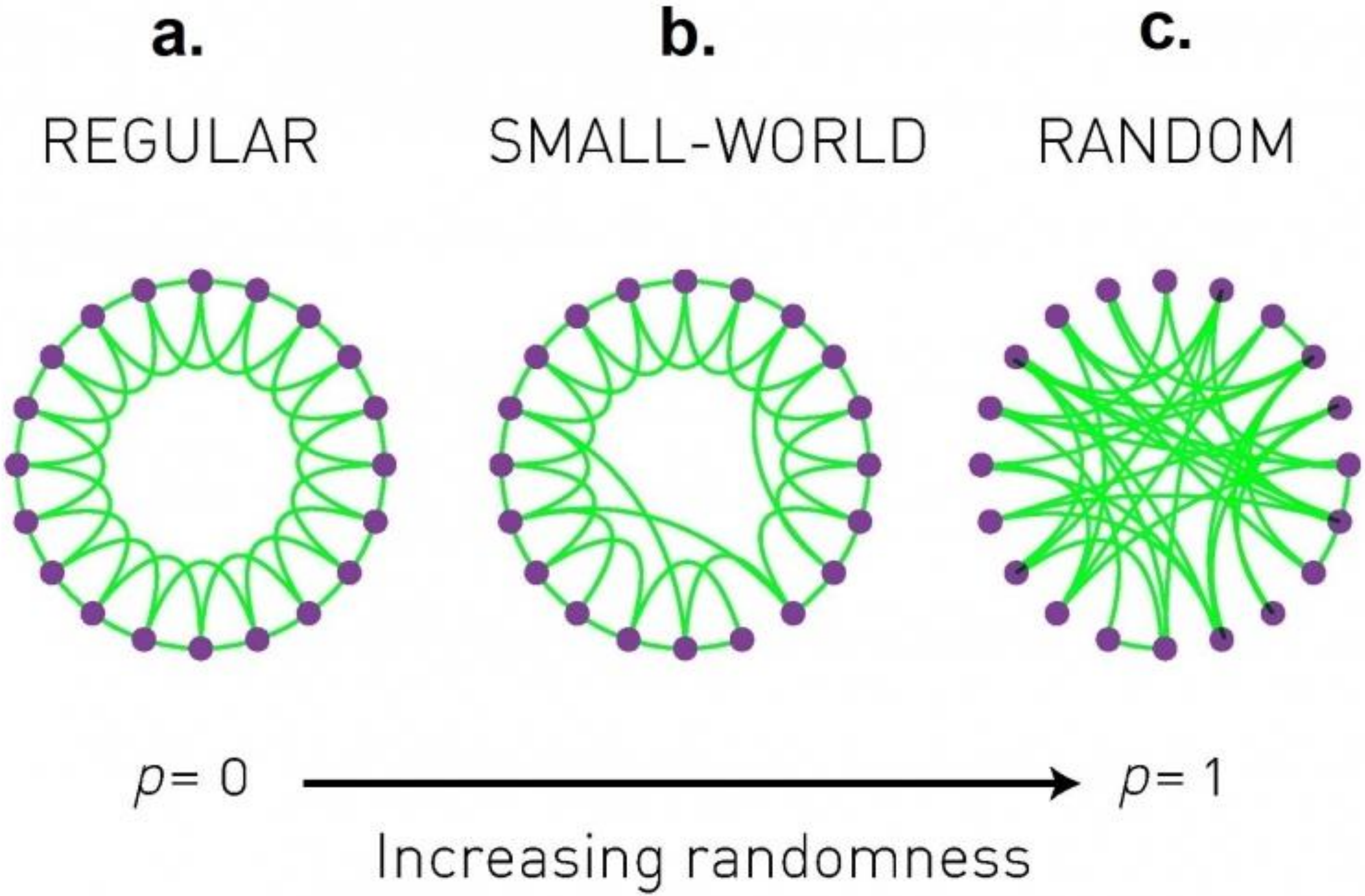
N=12, p=1/6

N=100, p=0.03

Small-World Model

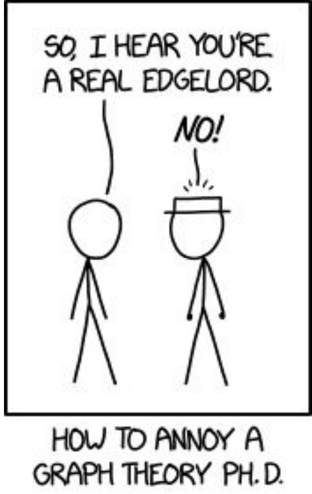
6 degrees of separation




a.
REGULAR
b.
SMALL-WORLD
c.
RANDOM
p= 0
p= 1
Increasing randomness

Conclusion

- Network science and the use of networks to represent, organize and analyze complex systems is becoming more and more popular.
- Many systems can be represented as a network, from transportation routes to social interactions to food webs.
- New tools to analyze complex networks are still being explored.



Questions?

References

- http://networksciencebook.com/
- https://en.wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg