Number Theory

Kayla Davie

Girls Talk Math 2019

June 27, 2018

What is Number Theory?

- Number theory is a branch of pure mathematics concerned with properties of the positive integers.
- Number theory is sometimes called “higher arithmetic.”
- Number theory has been studied since ancient times.
- Modern number theory is a broad subject that has been classified into subheadings which include:

  elementary number theory
  algebraic number theory
  analytic number theory
  geometric number theory
  probabilistic number theory

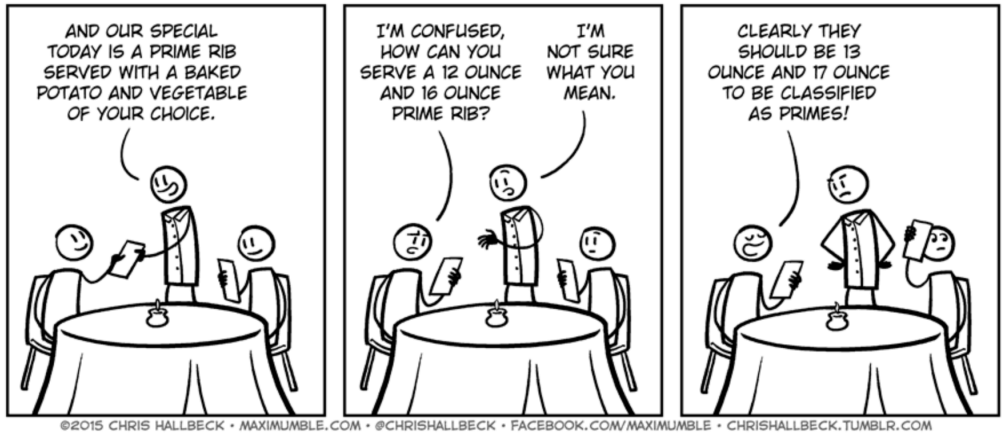
- “Mathematics is the queen of the sciences – and number theory is the queen of mathematics.”

  -Carl Friedrich Gauss (1777-1855)

What is a Prime Number?

- The set of integers is the set of all whole numbers, positive and negative
  {…, -2, -1, 0, 1, 2, …}.
- A prime number is a positive integer whose only factors are itself and 1. A number is composite if it is not prime.
- Unique Factorization Property for Primes: Every integer greater than 1 is either a prime number itself or can be represented as the product of prime numbers. Moreover, this representation is unique.

  This is also called the Fundamental Theorem of Arithmetic.



Divisibility

- Let n and m be two integers. We say a divides b if there exists a third integer q such that

  $$a \times q = b.$$

  Notation: $a|b$
  We say that b is a multiple of a (and q).
- The greatest common divisor (or gcd) of two integers a and b is the largest number d such that d divides a and d divides b.

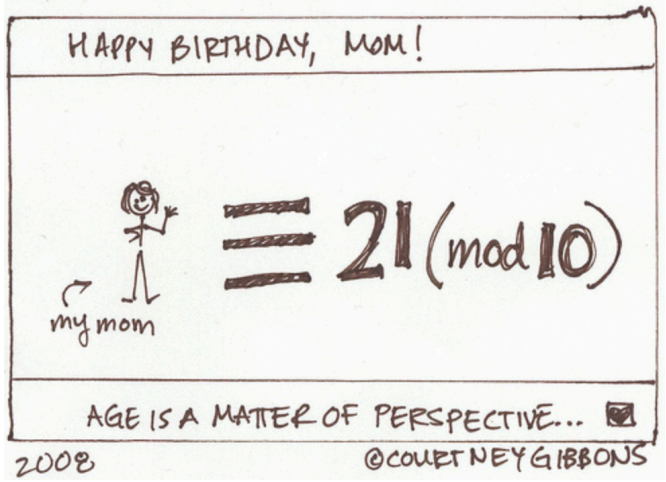
  $$d = \gcd(a, b)$$

  If the $\gcd(a, b) = 1$, then we say a and b are relatively prime.
- Ways to find the gcd of two integers:
  Listing all the factors of each integer and comparing
  Examining the prime factorizations of the two integers

  Example: Find gcd(36, 60).

Modular Arithmetic

- The Division Theorem: Suppose *a* and *b* are integers such that $b \neq 0$. Then there exists unique integers *q* and *r* such that
  $$a = b \times q + r.$$
  and such that $0 \leq r < |b|$.
- Modular Arithmetic: Let *a* and *b* be integers with $b \geq 0$. Then the number
  a mod b is the unique integer *r* such that
  $$a = b \times q + r$$
  with $0 \leq r \leq b - 1$.



Pythagoras

- Greek mathematician, 580-495 B.C.
- Worked in southern Italy among devoted followers, Pythagoreans
- His philosophy enshrined the number as the unifying concept necessary for understanding everything from planetary motion to musical harmony.

  one generator two opinion three harmony four justice

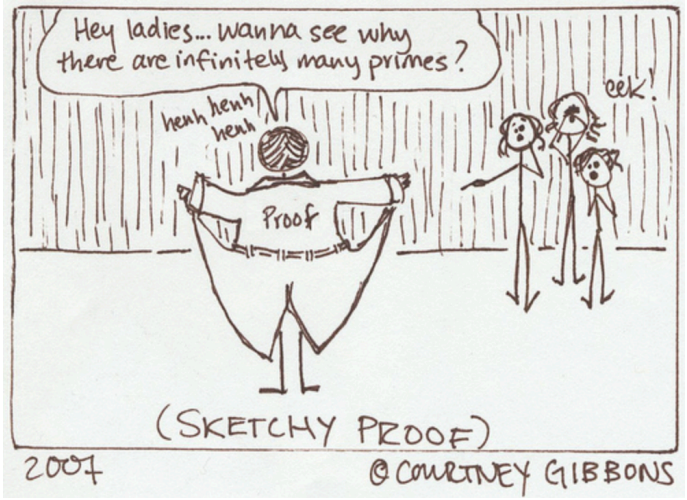
  five marriage six creation seven seven planets

- Controversial figure
- Perfect Number: A perfect number is a positive integer that is equal to the sum of its proper divisors.

Perfect Number	Positive Factors	Sum of all factors excluding itself
6	1, 2, 3, 6	6
28	1, 2, 4, 7, 14, 28	28
496	1, 2, 4, 8, 16, 31, 62, 124, 248, 496	496
8,128	1, 2, 4, 8, 16, 32, 64, 127, 254, 508, 1016, 2032, 4064, 8128	8,128

Euclid

- Greek mathematician born around mid-4th century B.C.
- "founder/father of geometry"
- Euclidean Algorithm: algorithm for finding the greatest common divisor of two whole numbers
- Unique Factorization Theorem (Fundamental Theorem of Arithmetic)
- Showed that no finite collection of primes contains them all.
- Provided a "recipe" for perfect numbers: if the series $1+2+4+8+\ldots+2^k$ sums to a prime, then the number $N= 2^k(1+2+4+\ldots+2^k)$ must be perfect.



Diophantus

- Green mathematician born around 200-215 B.C.
- “father of algebra”
- First Greek mathematician who recognized fractions as numbers
- Diophantine equations: algebraic equations with integer coefficients, for which integer solutions are sought
- Diophantus’ Riddle:

'Here lies Diophantus,' the wonder behold.
Through art algebraic, the stone tells how old:
'God gave him his boyhood one-sixth of his life,
One twelfth more as youth while whiskers grew rife;
And then yet one-seventh ere marriage begun;
In five years there came a bouncing new son.
Alas, the dear child of master and sage:
After attaining half the measure of his father's life chill fate took him. After consoling his fate by the science of numbers for four years, he ended his life.'

Prime Number Theorem

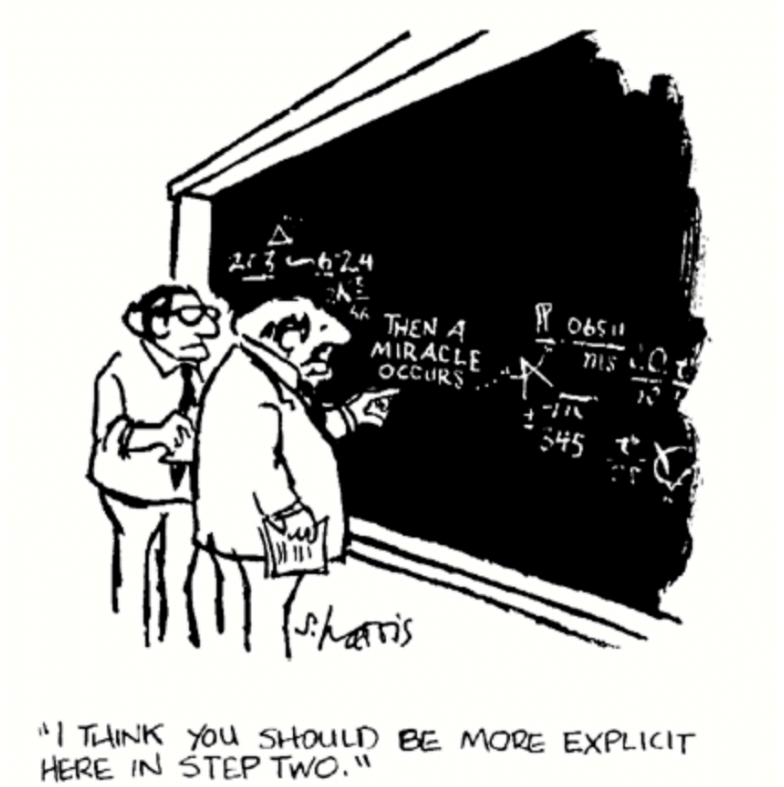
- describes the asymptotic distribution of prime numbers among the positive integers
- Prime Number Theorem (PNT): Let $\pi(x)$ be the prime-counting function that gives the number of primes less than or equal to x, for any real number x. The prime number theorem states that ${}^{x}/_{\log x}$ is a good approximation to $\pi(x)$*in* the sense that the limit of the quotient of the two functions $\pi(x)$ and ${}^{x}/_{\log x}$as x increases without bound is 1:

$$lim_{x\to\infty}\frac{\pi(x)}{\frac{x}{\log(x)}} = 1,$$

known as the asymptotic law of distribution of prime numbers.

n	$\pi(n) =$ **number of primes** $\leq n$	$\frac{\pi(n)}{n} =$ **proportion of primes among first** n **numbers**
10^2	25	0.25
10^4	$1,229$	0.1229
10^6	$78,498$	0.0785
10^8	$5,761,455$	0.0570
10^10	$455,052,511$	0.0455

Goldbach's Conjecture



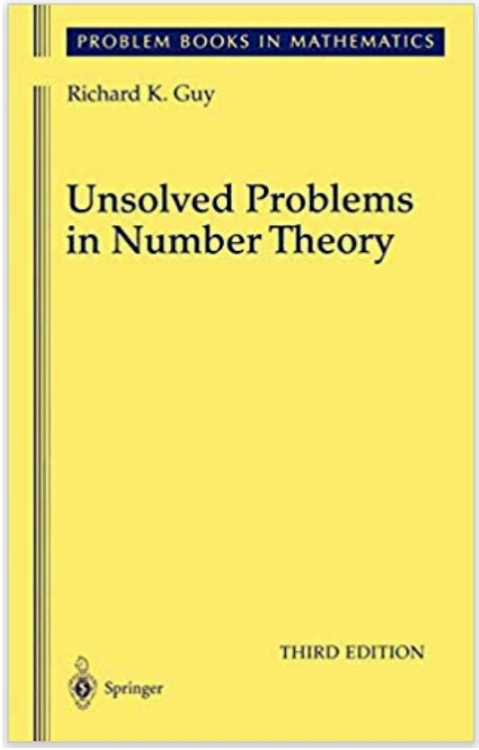
"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

- One of the oldest, best-known unsolved problems in number theory and all of mathematics.
- Goldbach's Conjecture: Every even integer greater than 2 can be expressed as the sum of two primes.
- Shown to hold for all even integers less than 4×10^{18}
- Weak Goldbach Conjecture: Every odd number greater than 5 can be expressed as the sum of three primes.
- Previously only shown to hold for odd integers greater than 2×10^{1346}

Conclusion

- Number theory is a fascinating field in pure mathematics.
- Several unsolved problems for YOU to solve!
- Some of the most brilliant minds in history were number theorists.

ARE YOU NEXT?!




Not to brag, but I just discovered the world's largest PRIME NUMBER.
It's more than 17 MILLION digits!
Enjoy 'em. They're the only digits you'll be getting tonight.
RJ
www.ITworld.com

Thank You!

References

- Dunham, William. “Number Theory.” Encyclopaedia Britannica. Encyclopaedia Brittanica, Inc., 2013. https://www.britannica.com/science/number-theory. Accessed: June 20, 2019.
- Sesiano, Jacques. “Diophantus.” Encyclopaedia Britannica. Encylclopaedia Brittanica, Inc., 2019. https://www.britannica.com/biography/Diophantus. Accessed: June 20, 2019.
- Thesleff, Holger. “Pythagoreanism.” Encyclopaedia Britannica. Encylclopaedia Brittanica, Inc., 2013. https://www.britannica.com/science/Pythagoreanism. Accessed: June 20, 2019.