The classification of surfaces

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Disclaimer

Keep in mind that in all that follows everything is thought of as being indefinitely flexible. This means that we can stretch, bend and inflate anything as much as we please, for example a lemon can be stretched out to be as big as the Sun.

Outline

1. Do surfaces exist?
   - What is a surface?
   - Concrete Examples
   - Exotic Examples
2. Setting the scene
   - Equivalence of surfaces
   - Different kinds of points
   - Two important properties
   - The connected sum
3. The classification
   - What do we want to do?
   - Statement of the classification

Overview

Some basic definitions

Definition

A geometric object is said to be **n-dimensional** if a creature moving inside it has n mutually perpendicular directions along which it can move at every point.

Definition

A 2-dimensional object is called a **surface**.

It often helps to think of a surface as being made out of small 2-dimensional patches.

Overview

Concrete Examples

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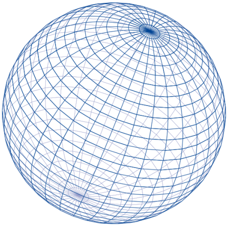
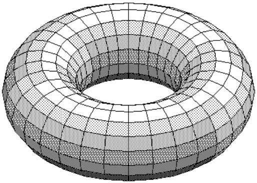
- The surface of a donut. This is basically everyone's favorite example
- The surface of a mug. (actually this might be everyone's favorite)

Concrete Examples

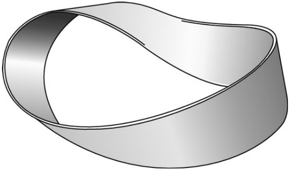




Overview

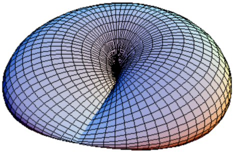
A mildly strange surface

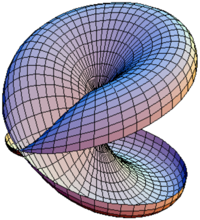
The Mobius band



A strange surface

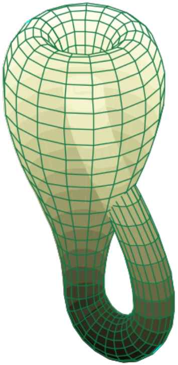
The projective plane





A very strange surface

The Klein bottle



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Equivalence of surfaces

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We say that two surfaces are **equivalent** if by deforming, stretching, twisting, squeezing and bending, but not cutting, tearing or gluing we can obtain one surface from the other. More formally, we say that the surfaces are **homeomorphic**.

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Surprisingly, and more famously, a mug and a donut are equivalent, as far as the theory of surfaces is concerned. Their surface is called a **torus**.

I'm not kidding, they're actually the same

Overview

Kinds of points

Definition

A point is said to be an **interior point** of a surface if it is contained in a disk which lies entirely on the surface.

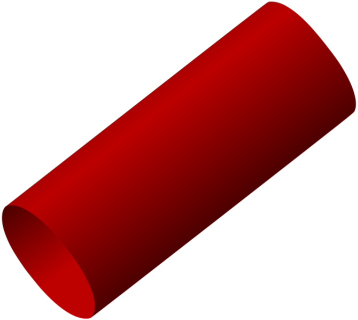
Definition

A point is said to be a **boundary point** of a surface if it is not an interior point and every open disk containing it also contains points outside of the surface.

The boundary of the surface is the set of all boundary points. As the name suggests, the boundary of a surface is just the edge of the surface .
For example, a sphere does not have a boundary, while the boundary of a disk is the circle enclosing it.

Example

Q: What is the boundary of this cylindrical surface?



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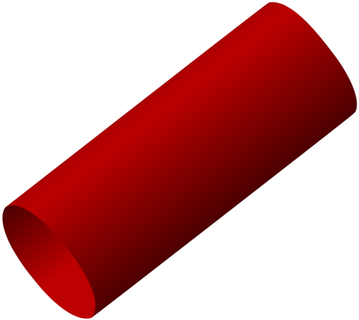

A: It consists of the two circular ends.

Overview

Compactness and Orientability

Definition

A surface is called **compact** if all of its boundary belongs to it and does not extend to infinity.

Definition

A surface is called **orientable** if it is possible to distinguish between its two faces.

For example, a torus has the inside and the outside face, while the plane has an upper and a lower face.

Not all surfaces are orientable!

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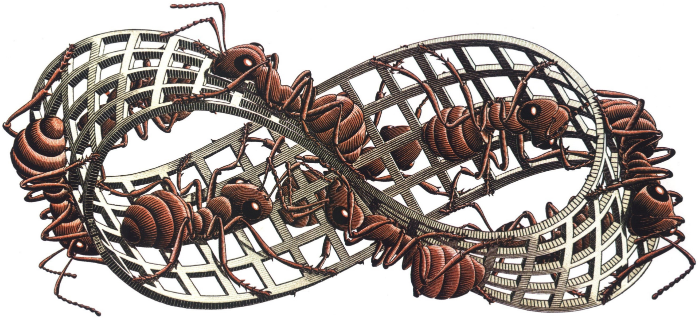
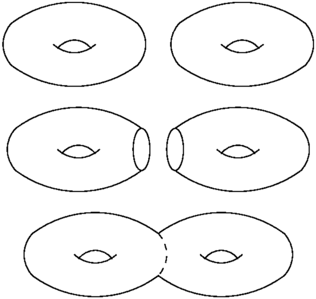
It's relatively easier to convince ourselves that the same is true for the Mobius band:



Overview

Connected sum

The **connected sum** of two surfaces is obtained by cutting off a small circular hole on each of them and then gluing along the cuts. For example:



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- We've seen that surfaces come in all sorts of unexpected shapes.
- We would like to write down a catalogue listing every possible surface.
- Unfortunately, this turns out to be very very hard.
- The good news is that we can (relatively) easily describe a very important class of surfaces.

Overview

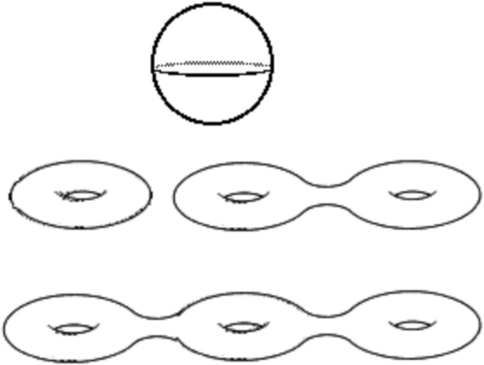
The classification

The **classification theorem** says that any compact surface with no boundary points is:

- a sphere or a connected sum of tori, if it is orientable, or
- a connected sum of projective spaces, if it is not orientable.

The bottom line

All compact surfaces without boundary that exist in our universe are orientable. Therefore, the classification theorem says that each and every one of them looks like one of the following:



(or potentially with more holes)

The end!